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Harmonic oscillator with potential barriers—exact solutions and perturbative treatments

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Abstract. Two specific potential-well problems have been solved exactly, firstly, a half-space harmonic oscillator with a finite-potential step and secondly, a full-space harmonic oscillator with an infinite wall located at a finite distance from the origin. The purpose of these studies is to investigate the limiting situations when (i) the finite-potential step in the first case is very high compared with the energy eigenvalues of the low-lying states and (ii) the infinite-potential wall in the second case is at a great distance from the origin where the amplitudes of the wavefunctions of the low-lying states are vanishingly small. The numerical results have been compared with the analytical expressions obtained from the perturbation schemes devised by Lee and Mei in previous work. Excellent agreements are obtained. General implications and applications are also discussed.

1. Introduction

Two novel perturbation schemes have recently been devised (Lee and Mei 1982). The first was used to treat the effects of a potential wall of great height V_0, starting from the unperturbed case where $V_0 \to \infty$, and the second was used to treat the effect of a remote infinite-potential barrier by constructing a pseudo-potential $\hat{H}_{ps}$ which is shown to reproduce the effects of the barrier. New procedures are then developed to handle the perturbative effects of $\hat{H}_{ps}$. In these cases, standard perturbation approaches are no longer valid, therefore, both schemes are unconventional.

These perturbation methods are very useful when handling the impurity states and exciton states with an additional boundary condition (ABC), such as an impurity centre or exciton confined in a semi-infinite or finite slab. The effects of the boundary are always difficult to treat in these cases. Levine (1965), Bastard (1981) and D'Andrea and Del Sole (1982) treated the effects of the surface using the variational method which is usually conjectural, the errors involved being difficult to estimate. Using these perturbative approaches the corrections to the eigenenergies and wavefunctions can be easily obtained.

The purpose of this work is to use the exact solutions of the well known problems to demonstrate the potential applications of these two perturbation schemes. Physicists have long known that one of the few exactly soluble problems is that of the harmonic oscillator (Schrödinger 1926) and it follows that the approximately soluble problems are those that can be studied by perturbing the oscillator. In §§ 2.1 and 2.2, we solve exactly two specific potential-well problems associated with harmonic oscillators: firstly, a half-space harmonic oscillator with a finite-potential step and secondly, a full-space harmonic oscillator with an infinite wall located at a finite distance from the origin. The numerical results of the above problems, in the limiting situations when the finite step is very high compared with the eigenenergies of the low-lying states and the infinite wall is at a great distance from the origin where the wavefunctions of the low-lying states are almost vanished, are compared with the analytical expressions obtained from the perturbation schemes in § 3 and excellent agreements are obtained. Further remarks and discussions are included in the last section.

2. Exact solutions

2.1. Half-space harmonic oscillator with a finite potential step

The Schrödinger equations for the potential $V(x)$ shown in figure 1 are given by

$$\begin{aligned} -\frac{\hbar^2}{2\mu}\frac{d^2\Psi_R}{dx^2}+(\tfrac{1}{2}\mu\omega^2x^2-E)\Psi_R&=0 \qquad x>0 \\ -\frac{\hbar^2}{2\mu}\frac{d^2\Psi_L}{dx^2}+(V_0-E)\Psi_L&=0 \qquad x<0 \end{aligned} \tag{1}$$

where μ is the mass of the particle, ω the classical frequency of the harmonic oscillator, V_0 the height of the potential wall, E the eigenenergy and Ψ_R and Ψ_L the wavefunctions in the regions $x>0$ and $x<0$.

In the regions where $x<0$, the wavefunctions can be easily obtained:

$$\Psi_L = A\,e^{kx}$$

where

$$k=\{[2\mu(V_0-E)]/\hbar^2\}^{1/2} \tag{2}$$

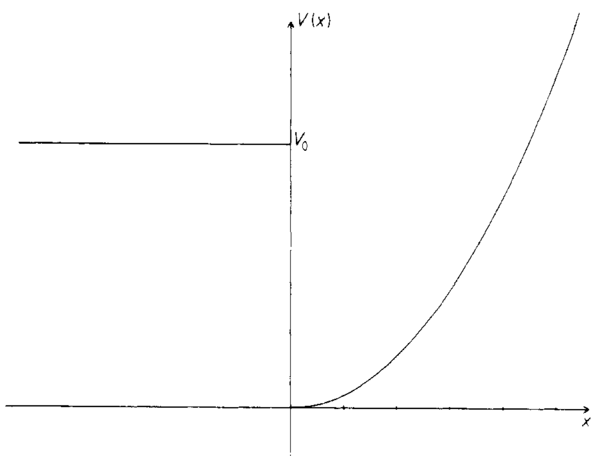

Figure 1. One-dimensional half-space harmonic-oscillator potential well with a finite potential step V_0.

and A is the normalisation constant of the wavefunction. We only adopt the decaying solution in order to justify the asymptotic behaviour requirement as $x \to -\infty$. As $x > 0$, we have

$$\frac{d^2\Psi_R}{dx^2} + \left(\frac{2\mu E}{\hbar^2} - \frac{\mu^2\omega^2}{\hbar^2}x^2\right)\Psi_R = 0. \tag{3}$$

Let

$$z \equiv (2\mu\omega/\hbar)^{1/2}x \equiv \sqrt{2\alpha}x; \tag{4}$$

then

$$\frac{d^2\Psi_R}{dz^2} + \left(\frac{E}{\hbar\omega} - \tfrac{1}{4}z^2\right)\Psi_R = 0. \tag{5}$$

Compare equation (5) with the Weber equation (Morse and Feshbach 1953)

$$\frac{d^2\Psi}{dz^2} + (m + \tfrac{1}{2} - \tfrac{1}{4}z^2)\Psi = 0; \tag{6}$$

correspondingly, we have

$$E/\hbar\omega = m + \tfrac{1}{2}. \tag{7}$$

The solution of equation (5) is then the well known Weber function

$$D_m(z) = 2^{m/2}\,e^{-z^2/4}\left(\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-\frac{1}{2}m)}F(-\tfrac{1}{2}m|\tfrac{1}{2}|\tfrac{1}{2}z^2) - \frac{\sqrt{2\pi}z}{\Gamma(-\frac{1}{2}m)}F(\tfrac{1}{2}-\tfrac{1}{2}m|\tfrac{3}{2}|\tfrac{1}{2}z^2)\right) \tag{8}$$

where $F(a|b|x)$ is the confluent hypergeometric function (Abramowitz and Stegun 1964) defined as

$$F(a|b|x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{2!\,b(b+1)}x^2 + \frac{a(a+1)(a+2)}{3!\,b(b+1)(b+2)}x^3 + \ldots \tag{9}$$

and $\Gamma(z)$ is the gamma function (Abramowitz and Stegun 1964) defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}\,e^{-t}\,dt \qquad \text{for } \mathrm{Re}\, z > 0. \tag{10}$$

The recursion relations for the Weber function $D_m(z)$ are

$$\begin{aligned}
&D_{m+1}(z) - zD_m(z) + mD_{m-1}(z) = 0 \\
&(d/dz)D_m(z) - \tfrac{1}{2}zD_m(z) + D_{m+1}(z) = 0 \\
&(d/dz)D_m(z) + \tfrac{1}{2}zD_m(z) - mD_{m-1}(z) = 0.
\end{aligned} \tag{11}$$

In general, m is not an integer.

The function $D_m(z)$ is chosen to vanish as $z \to \infty$, but not usually as $z \to -\infty$. The asymptotic behaviour of $D_m(z)$ is given by

$$D_m(\sqrt{2\alpha}x) \to \begin{cases} (\sqrt{2\alpha}x)^m \exp(-\tfrac{1}{2}\alpha x^2) & x \to \infty \\ \dfrac{\sqrt{2\pi}}{\Gamma(-m)}\dfrac{\exp(\tfrac{1}{2}\alpha x^2)}{(-\sqrt{2\alpha}x)^{m+1}} & x \to -\infty \end{cases} \tag{12}$$

unless m is zero or a positive integer, in which case

$$D_m(z)=(-1)^m D_m(-z). \tag{13}$$

In order to have a wavefunction which is quadratically integrable, we must have $m=0,1,2\ldots$, which limits the energy levels of the linear harmonic oscillator to discrete allowed values

$$E_m=(m+\tfrac{1}{2})\hbar\omega. \tag{14}$$

When m is a positive integer, the Weber function becomes proportional to the Hermite polynomials

$$D_m(z)=2^{m/2}\exp(-\tfrac{1}{4}z^2)H_m(z/\sqrt{2}) \tag{15}$$

where $H_m(z)$ are the Hermite polynomials.

Then we can easily see that the Weber function (8) satisfies the asymptotic behaviour of this problem at both $x\to\pm\infty$. The next step is to match the logarithmic derivatives of the wavefunctions Ψ_R and Ψ_L at $x=0$.

In the region $x<0$, the logarithmic derivative is

$$(\Psi_L'(x)/\Psi_L(x))|_{x=0}=\{[2\mu(V_0-E)]/\hbar^2\}^{1/2}. \tag{16}$$

In the region $x>0$, making use of the recursion relations of the Weber functions equation (11), we have

$$\begin{aligned}\left.\frac{\Psi_R'(x)}{\Psi_R(x)}\right|_{x=0}&=-\sqrt{2\alpha}\frac{2^{(m+1)/2}(\sqrt{\pi}/\Gamma(\frac{1}{2}-\frac{1}{2}(m+1)))}{2^{m/2}(\sqrt{\pi}/\Gamma(\frac{1}{2}-\frac{1}{2}m))}\\&=-2\sqrt{\alpha}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}m)}{\Gamma(-\frac{1}{2}m)}.\end{aligned} \tag{17}$$

Then the energy eigenvalues can be determined from the equation

$$[\tfrac{1}{2}(v_0-m-\tfrac{1}{2})]^{1/2}=-\frac{\Gamma(\frac{1}{2}-\frac{1}{2}m)}{\Gamma(-\frac{1}{2}m)}$$

where

$$v_0=V_0/\hbar\omega. \tag{18}$$

In order to solve equation (18) we have to use the identity of the gamma function

$$-z\Gamma(z)\Gamma(-z)=\pi(\sin\pi z)^{-1}. \tag{19}$$

Let

$$g(m)=\frac{\Gamma(\frac{1}{2}-\frac{1}{2}m)}{\Gamma(-\frac{1}{2}m)}. \tag{20}$$

The numerical solutions of m can be obtained if the value of v_0 is given. We have plotted the graphic solutions of m in figure 2 for several values of v_0. It is obvious that as $v_0\to\infty$, the only possible solutions for m are the odd positive integers, which is exactly the case for a half-space harmonic oscillator with an infinite-potential wall.

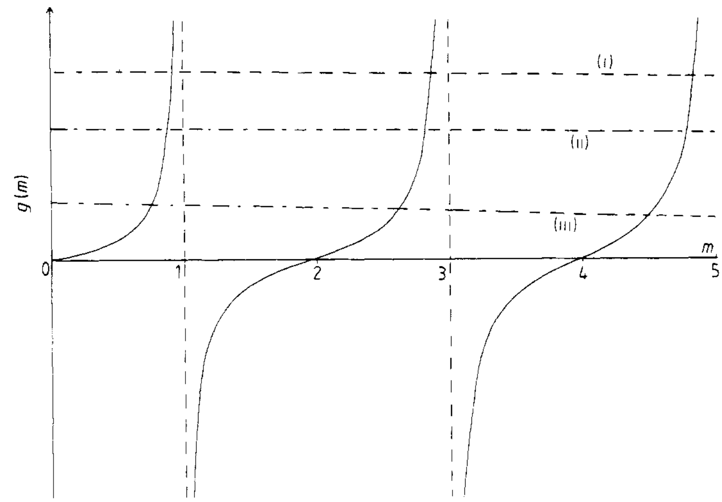

Figure 2. The graphic solutions of equation (18) for different values of v_0 where (i) $v_0 = 150$, (ii) $v_0 = 50$ and (iii) $v_0 = 10$.

2.2. Full-space harmonic oscillator with an infinite wall located at a finite distance from the origin

Now let us consider the Schrödinger equation for the potential $V(x)$ shown in figure 3:

$$\begin{aligned} &-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}+(\tfrac{1}{2}\mu\omega^2x^2-E)\Psi=0 \qquad && x>-r \\ &\Psi(x)=0 && x<-r. \end{aligned} \tag{21}$$

The solution for the wavefunction Ψ in the region $x>-r$ is still the Weber function

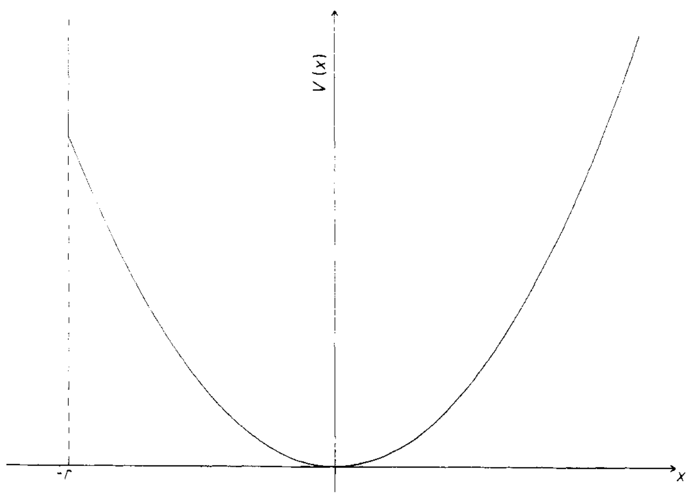

Figure 3. One-dimensional full-space harmonic-oscillator potential well with an infinite wall located at a distance r from the origin.

equation (8). In order to satisfy the boundary condition at $x = -r$, however, we require

$$\frac{F(-\frac{1}{2}m|\frac{1}{2}|\frac{1}{2}R^2)}{\Gamma(\frac{1}{2}-\frac{1}{2}m)} = -\frac{\sqrt{2}RF(\frac{1}{2}-\frac{1}{2}m|\frac{3}{2}|\frac{1}{2}R^2)}{\Gamma(-\frac{1}{2}m)} \tag{22}$$

where

$$R = (2\mu\omega/\hbar)^{1/2}r.$$

Solving the above equation for given values of R, we can determine the eigenvalues of this potential-well problem. As the infinite wall is located on the right-hand side, $D_m(-z)$ is adopted as the wavefunction in order to obtain the correct asymptotic behaviour as $x \to \pm\infty$. The final results, however, remain the same.

As $R \to 0$, we can easily see that the only possible solutions are m to be the odd positive integers, which is exactly the case for the half-space harmonic oscillator. As $R \to \infty$, we have to use the asymptotic expressions for the confluent hypergeometric functions. Since

$$F(a|b|z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}[1+O(1/z)] \qquad \text{for Re } z > 0 \tag{23}$$

we define

$$f(m, R) = \frac{F(-\frac{1}{2}m|\frac{1}{2}|\frac{1}{2}R^2)}{\Gamma(\frac{1}{2}-\frac{1}{2}m)} + \frac{\sqrt{2}RF(\frac{1}{2}-\frac{1}{2}m|\frac{3}{2}|\frac{1}{2}R^2)}{\Gamma(-\frac{1}{2}m)}. \tag{24}$$

Then, as R is very large, the asymptotic expression for $f(m, R)$ is given by

$$f(m, R) \simeq \left(\frac{\exp(\frac{1}{2}R^2)\Gamma(\frac{1}{2})}{(\frac{1}{2}R^2)^{(1+m)/2}}\right)\left(\frac{1}{\Gamma(-\frac{1}{2}m)\Gamma(\frac{1}{2}-\frac{1}{2}m)}\right) \qquad \text{as } R \to \infty. \tag{25}$$

Therefore, the values of m have to be positive integers in order to satisfy the equation $f(m, R) = 0$. This reduces to the case of a full-space harmonic oscillator. The numerical solutions of m can be obtained for a given R and are plotted in figure 4.

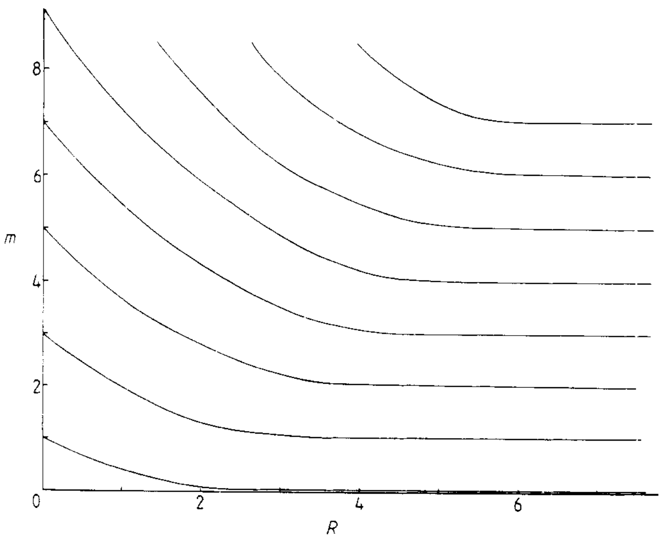

Figure 4. The graphical solutions of equation (22) as a function of R.

3. Perturbative treatments

In the first case, as the height of the finite step V_0 is very large compared with the eigenvalues of the low-lying states, the exact solutions are very close to those for the case $V_0 \to \infty$. The difference, as mentioned by Lee and Mei (1982), is proportional to δ/a, where $\delta \simeq [\hbar^2/2\mu(V_0-E)]^{1/2}$ corresponds to the penetration depth of the wavefunction into the finite-potential wall. Here a is the characteristic size and E the eigenenergy of the hard-welled eigenfunctions.

The unperturbed eigenfunctions $\psi_n^{(0)}(x)$ and eigenvalues $E_n^{(0)}$ corresponding to the limiting case $V_0 \to \infty$ are well known, being just those associated with the odd-parity solutions for the harmonic oscillator in full space, with no wall. According to the previous work we replace the infinite wall at $x=0$ with an infinite one at $x=-\delta^{(n)}$. The equivalent hard-well problem has

$$V(x)=\begin{cases}\frac{1}{2}\mu\omega^2x^2 & x>-\delta^{(0)}\\ \infty & x<-\delta^{(0)}\end{cases} \tag{26}$$

with

$$\delta^{(0)}=[\hbar^2/2\mu(V_0-E^{(0)})]^{1/2} \tag{27}$$

where $E^{(0)}$ is the unperturbed energy eigenvalue.

To relate this problem to the unperturbed case we perform a change of variable from x to $y=x+\delta^{(0)}$. The equivalent problem then becomes

$$\begin{aligned}&\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\tfrac{1}{2}\mu\omega^2(y-\delta^{(0)})^2\right)\psi(y)=E(y) && y>0\\ &\psi(y)=0 && y\leqslant 0.\end{aligned} \tag{28}$$

Obviously, we may now divide the Hamiltonian, in the new variable y, into the unperturbed part H_0 and the perturbation term H':

$$H=H_0+H' \qquad H_0=-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\tfrac{1}{2}\mu\omega^2y^2 \qquad H'=-\mu\omega^2\delta^{(0)}y+\mathrm{O}[(\delta^{(0)})^2]. \tag{29}$$

Note that the term H' itself depends, through $\delta^{(0)}$, on the unperturbed energy. The energy corrections ΔE_n due to the finiteness of the wall V_0 at $x=0$ can be obtained immediately from

$$\Delta E_n=\int_0^\infty \psi_n^{*(0)}(y)H'\psi_n^{(0)}(y)\,\mathrm{d}y=-\mu\omega^2\delta^{(0)}\langle y\rangle\equiv-\gamma_nE_n^{(0)}/(v_n)^{1/2} \tag{30}$$

where

$$\langle y\rangle\equiv\int_0^\infty \psi_n^{*(0)}(y)y\psi_n^{(0)}(y)\,\mathrm{d}y \tag{31}$$

$$E_n^{(0)}=(n+\tfrac{1}{2})\hbar\omega \qquad n=1,3,5 \tag{32}$$

$$v_n=(V_0-E_n^{(0)})/\hbar\omega \tag{33}$$

and the first six values of γ_n are

$$\begin{aligned}&\gamma_1=\tfrac{4}{3}(2\pi)^{-1/2} && \gamma_3=\tfrac{6}{7}(2\pi)^{-1/2} && \gamma_5=\tfrac{15}{22}(2\pi)^{-1/2}\\ &\gamma_7=\tfrac{7}{12}(2\pi)^{-1/2} && \gamma_9=\tfrac{315}{608}(2\pi)^{-1/2} && \gamma_{11}=\tfrac{693}{1472}(2\pi)^{-1/2}.\end{aligned} \tag{34}$$

We can easily observe that the energy corrections are proportional to the spatial spreads of the wavefunctions. As expected from the variational principle, the energy corrections are all negative. We also solved equation (18) numerically for different values of v_0 and tabulated the comparisons with the γ_n obtained from equation (34) in table 1. We found excellent agreement for a wide range of values of v_0.

In the second case, when the infinite wall is a great distance from the centre of the harmonic oscillator well, where the amplitudes of the low-lying state wavefunctions are vanishingly small, it is clear physically that the exact solution is very close to that obtained when the infinite potential wall is infinitely far away from the origin. It is not easy, however, to find the difference between these two.

In our previous work (Lee and Mei 1982) we proposed a pseudo-potential $\hat{H}_{ps}$ where

$$\hat{H}_{ps} = (\hbar^2/2\mu)\delta(x+r)(\mathrm{d}/\mathrm{d}x)_{x=-r+\varepsilon} \tag{35}$$

Table 1. Comparison of the exact numerical solutions from equation (18) and the perturbation results from equation (34) on the energy correction of the half-space harmonic oscillator with a finite potential step V_0, where $v_0 \equiv V_0/\hbar\omega$.

v_0	γ_n (numerical results)	γ_n (perturbation results)
1×10^{10}	0.531 922	0.531 923
	0.341 950	0.341 951
	0.272 006	0.272 006
	0.232 716	0.232 716
	0.206 689	0.206 689
	0.187 817	0.187 817
1×10^{8}	0.531 910	0.531 923
	0.341 945	0.341 951
	0.272 002	0.272 006
	0.232 713	0.232 716
	0.206 687	0.206 689
	0.187 815	0.187 817
1×10^{6}	0.531 793	0.531 923
	0.341 893	0.341 951
	0.271 970	0.272 006
	0.232 690	0.232 716
	0.206 668	0.206 689
	0.187 800	0.187 817
1×10^{4}	0.530 594	0.531 923
	0.341 337	0.341 951
	0.271 589	0.272 006
	0.232 389	0.232 716
	0.206 410	0.206 689
	0.187 569	0.187 817
1×10^{2}	0.516 244	0.531 923
	0.332 146	0.341 951
	0.263 209	0.272 006
	0.223 990	0.232 716
	0.197 719	0.206 689
	0.178 471	0.187 817

and showed that $\hat{H}_{ps}$ reproduces the effect of the infinite barrier. That is, if we treat literally the pseudo-potential together with the original problem it is equivalent to solving exactly the Schrödinger equation with its boundary condition. But in the case when $d \gg \alpha^{-1/2}$, that is when the infinite wall is far away from the important region of the potential well, we can perform perturbative calculation on the energy corrections by using the technique developed in the previous work (Lee and Mei 1982). This perturbation scheme is not the same as the conventional method as we show that the zeroth-order wavefunction and the perturbed wavefunction are of the same order of magnitude near the vicinity of the infinite wall. Therefore, the energy correction is shown to be

$$\Delta E_n \simeq 2\langle\psi_n^{(0)}|\hat{H}_{ps}|\psi_n^{(0)}\rangle. \tag{36}$$

The important difference between this perturbation method and the ordinary perturbation method is that we also have to include the expectation value of the perturbed Hamiltonian on the perturbation wavefunction, which is considered to be negligible in ordinary perturbation schemes. Thus, we have calculated the first four energy corrections ΔE_n for the harmonic oscillator potential.

Since we have

$$E_n = E_n^{(0)} + \Delta E_n \tag{37}$$

where $E_n^{(0)} = (n+\frac{1}{2})\hbar\omega$ and $\Delta E_n = \kappa_n\hbar\omega$, $n = 0, 1, 2, 3 \ldots$ then

$$\begin{aligned} &\kappa_0 = \tfrac{1}{2}(2\pi)^{-1/2}R\,\mathrm{e}^{-R^2/2} \qquad \kappa_1 = \tfrac{1}{2}(2\pi)^{-1/2}R(R^2-2)\,\mathrm{e}^{-R^2/2} \\ &\kappa_2 = \tfrac{1}{4}(2\pi)^{-1/2}(R^2-1)(R^3-5R)\,\mathrm{e}^{-R^2/2} \\ &\kappa_3 = \tfrac{1}{12}(2\pi)^{-1/2}(R^3-3R)(R^4-9R^2+6)\,\mathrm{e}^{-R^2/2}. \end{aligned} \tag{38}$$

The comparison with the exact numerical results is in table 2. The agreements are good for large values of R. This is because in arriving at equation (36) we have assumed that the wavefunction decays exponentially away from the important region, whereas in this case the wavefunctions are enveloped by a Gaussian function. Therefore, we observe that the perturbation results are slightly overestimated. The agreements improve as the distance R gets larger.

4. Discussion

We have solved two specific potential problems exactly, (i) a half-space harmonic oscillator with a finite potential step and (ii) a full-space harmonic oscillator with an infinite potential wall located at a finite distance from the origin. Then we used the perturbation schemes developed in previous work and demonstrated the excellent agreement between the exact results and the perturbation calculations when (i) the finite potential step is very high compared with the eigenenergies of the low-lying states and (ii) the infinite potential wall is very far away from the essential region of the potential well.

Even though all the calculations performed in this paper are one dimensional, the methods themselves are very general in nature and can be extended easily to the three-dimensional case.

Table 2. Comparison of the exact numerical solutions and the perturbation results on the energy correction of the full-space harmonic oscillator with an infinite potential wall located at a distance r from the origin, where $R = \sqrt{2\alpha}r$. In each block the first number is the exact numerical result from equation (22), the second is the perturbation result from equation (38) and the third is the ratio of these two numbers.

R	$m = 0$	$m = 1$	$m = 2$	$m = 3$
4.2	$2.312\,180 \times 10^{-4}$ $2.475\,008 \times 10^{-4}$ (0.9340)			
4.8	$1.809\,315 \times 10^{-5}$ $1.901\,423 \times 10^{-5}$ (0.9516)	$3.733\,496 \times 10^{-4}$ $4.000\,595 \times 10^{-4}$ (0.9332)		
5.4	$9.656\,544 \times 10^{-7}$ $1.002\,975 \times 10^{-6}$ (0.9628)	$2.595\,084 \times 10^{-5}$ $2.724\,081 \times 10^{-5}$ (0.9526)	$3.195\,946 \times 10^{-4}$ $3.411\,848 \times 10^{-4}$ (0.9367)	
6.0	$3.537\,631 \times 10^{-8}$ $3.645\,528 \times 10^{-8}$ (0.9704)	$1.195\,248 \times 10^{-6}$ $1.239\,480 \times 10^{-6}$ (0.9643)	$1.890\,065 \times 10^{-5}$ $1.977\,699 \times 10^{-5}$ (0.9557)	$1.848\,663 \times 10^{-4}$ $1.960\,930 \times 10^{-4}$ (0.9427)
6.6	$8.930\,985 \times 10^{-10}$ $9.152\,086 \times 10^{-10}$ (0.9758)	$3.696\,796 \times 10^{-8}$ $3.803\,606 \times 10^{-8}$ (0.9719)	$7.260\,291 \times 10^{-7}$ $7.509\,808 \times 10^{-7}$ (0.9668)	$8.974\,726 \times 10^{-6}$ $9.350\,954 \times 10^{-6}$ (0.9598)
7.2		$7.742\,837 \times 10^{-10}$ $7.923\,326 \times 10^{-10}$ (0.9772)	$1.843\,487 \times 10^{-8}$ $1.892\,875 \times 10^{-8}$ (0.9739)	$2.794\,300 \times 10^{-7}$ $2.881\,643 \times 10^{-7}$ (0.9697)
7.8			$3.128\,872 \times 10^{-10}$ $3.196\,490 \times 10^{-10}$ (0.9788)	$5.688\,793 \times 10^{-9}$ $5.828\,048 \times 10^{-9}$ (0.9761)
8.4				$7.666\,204 \times 10^{-11}$ $7.821\,230 \times 10^{-11}$ (0.9802)

The purpose of this work is to demonstrate the potential applications of the perturbation schemes we have developed. Our aims are to treat a general class of problems and provide a check for any specific problem involving a barrier solved by other approximation means such as the variational method.

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